S. F. Grace

I. Historical Review
of Dynamical Explanations of Tides
in Non-Elongated Enclosed Seas and Lakes

II. Historical Review
of Dynamical Explanations of the Tides
of the Mediterranean, the Baltic Sea,
the Gulf of Mexico and the Arctic Ocean
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By S. F. GRACE
(University of Liverpool)

§ 1. The following review of the tides in non-elongated enclosed basins relates to publications involving dynamical explanations of tidal motion either in actual seas and lakes or in geometrically simple basins; it may be regarded as a companion to a preceding one on tides and seiches in narrow seas and lakes. The publications are given mainly under groups A and B in the bibliography of Marees, published in these Bulletins, and to which reference is made. Explanations relating to the Mediterranean, the Baltic Sea and the Gulf of Mexico have not been included in the discussion although these basins are practically land-locked and for some purposes can be regarded as closed; publications in connection with these basins and the Arctic Ocean will be discussed in a further review.

Broadly speaking, the present review indicates the developments of two modes of discussion. One concerns the theoretical explanation of actual tides in non-elongated enclosed basins, either by the equilibrium theory or by application of narrow-sea methods, the other concerns the hydrodynamical theory of tidal motion in geometrically simple basins; it is only within recent years that hydrodynamical theory has advanced sufficiently to be able to indicate agreement between exact theory and observation.

In the review itself symbols will rarely be used; the narrow-sea methods have already been explained in the preceding review.

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and the analysis for the exact theory is in many cases so heavy that it is found almost impossible to give details and at the same time to restrict the review within reasonable limits. The problem involved in the exact theory is, however, indicated in the following section.

§ 2. In order to explain the hydrodynamical problem of the tides in small enclosed basins, the following notation is used:

- \( x, y \) = the rectangular coordinates of a point in the mean surface of the water,
- \( h \) = the depth of water below any point of the mean surface,
- \( \zeta \) = the elevation of the free surface above any point of the mean surface,
- \(-g\zeta\) = the potential of the external disturbing forces,
- \( \omega \) = the angular speed of rotation of the basin about the vertical,
- \( v \) = the speed of a simple harmonic oscillation of the water.

Also let \( \partial / \partial n \) denote differentiation in the direction of the outward normal to a boundary of the basin and \( \partial / \partial x \) in the direction a right angle in advance and therefore tangential to the boundary.

The equations defining the periodic oscillations of water in a rotating basin are, writing \( \zeta' = \zeta - \zeta' \),

\[
\frac{\partial}{\partial t} \left( \frac{\partial \zeta'}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \zeta'}{\partial y} \right) + 2\omega \left( \frac{\partial \zeta'}{\partial x} \right) + \frac{\partial^2 - 4\omega^2}{g} \zeta' = 0, \tag{2.1}
\]

over the surface, and

\[
\hat{A} \left( i\zeta' \frac{\partial^2}{\partial x^2} + 2\omega \frac{\partial^2}{\partial x \partial y} \right) = 0, \tag{2.2}
\]

at the boundary.

The problems of tidal motion in geometrically simple basins, which appear in the review, are mainly dependent on these equations. They fall into various classes:

1. Free oscillations in a non-rotating basin; here \( \omega = 0 \), \( \zeta' = 0 \).
2. Free oscillations in a rotating basin; here \( \zeta' = 0 \).
3. Forced oscillations in a non-rotating basin; here \( \omega = 0 \).
4. Forced oscillations in a rotating basin.

In (1) and (2) \( 2\pi/\omega \) is one of the free periods, whereas in (3) and (4) \( 2\pi/\omega \) is the period of the disturbing force.

§ 3. D. Bernoulli was the first to attempt to compute the tides of an inland sea. In his "Traité sur le Flux et le Reflux de la Mer" he considered a small enclosed sea on the equator and gave a rule comparing the range of the tide with that for the earth covered with water, calculated according to the equilibrium theory, the development of which he had previously given; he also noted the necessity for a correction to the equilibrium theory in the case of a small sea. The maximum height of the spring tides in the Caspian Sea was only 1.5 inches, since, in addition to the introduction of the factor \( \chi/2 \), he assumed a different value from that of Bernoulli for the range on an earth covered with water.

The free oscillations of a non-rotating circular sea of uniform depth were discussed by S. D. Poisson \(^2\) though he was unable to interpret his results. He also gave the solution of the problem of the transverse vibrations of a uniform rectangular membrane \(^3\), the equation for which is identical in form with that for the free oscillations of a non-rotating rectangular sea of uniform depth but the boundary conditions differ in the two cases.

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5. *L'ouvrage du Journal, 13, 1869; Cours de physique mathematique, 122, 1873.
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\[ \frac{\partial}{\partial x} \left( \frac{\partial \zeta'}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \zeta'}{\partial y} \right) + \frac{2v}{\partial x} \left( \frac{\partial \zeta'}{\partial y} \right) + \frac{2v}{\partial y} \left( \frac{\partial \zeta'}{\partial x} \right) - \frac{g}{\omega^2} \zeta' = 0, \tag{2.1} \]

over the surface, and

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3 Sull'ellittica delle Memorie della Memoria del Corso di Poesia, 1823-4.
5 Liouville's Journal, 12, 1868; Cours de physique mathématique, 122, 1873.
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W. Ferrel compared the observed M2 tide at Chicago with that obtained for a uniform canal, of the same length and depth as Lake Michigan, extending along a meridian but the amplitude in the canal was found to be too small.

The free oscillations of a non-rotating circular sea of uniform depth were considered with numerical details by Lord Rayleigh 1, the solution involved Bessel functions, the analysis for which had not been developed in Poisson's time. The case of a circular sectorial sea he considered in his Theory of Sounds 2.

The first attempt to obtain numerical results in respect of the effect of the earth's rotation on the oscillations of enclosed seas was made by Sir W. Thomson (Lord Kelvin) when he applied the dynamical theory of tides to small seas. He considered the case of the free oscillations of a rotating circular sea of uniform depth, using equation (2.1) with \( \alpha = 0 \), expressed in terms of polar coordinates; he gave the solution of his equation in normal form.

The free oscillations in a non-rotating basin in the form of a hyperboloid of revolution or a cone were given by A. G. Greenhill. 3

In his treatise on Hydrodynamics, H. Lamb considered the free and forced oscillations of a rotating circular sea (a) for which the depth is uniform and (b) for which the basin is a paraboloid of revolution, the free oscillations of a circular sea of uniform depth being given in fuller detail than by Thomson. He noted the existence in case (b) of free oscillations of the second class, which have speeds tending to vanish with the speed of rotation; free oscillations of the first class tend to become those of a non-rotating sea as the speed of rotation tends to zero.

R. A. Harris 4 applied the equilibrium theory to determine the semi-diurnal tides in certain seas and lakes. At Duluth on Lake Superior he obtained good agreement with observation, but at Chicago and Milwaukee on Lake Michigan the amplitudes were considerably smaller than the observed values and the phases were not in agreement.

G. H. Darwin suggested that the tides in a lake of not too large extent would be determined by the equilibrium theory. It was pointed out by Harris, however, that the phases of the tides of Lake Michigan are better accounted for by the canal theory than by the equilibrium theory but he expressed the opinion that the latter theory would account well for the tides of the Black Sea.

Rayleigh, considering the effect of slow rotation on the free oscillations of systems, showed that to every free mode in a non-rotating basin there corresponds a free mode in the slowly rotating basin. In a subsequent paper he solved the problem of the free oscillations of a rotating rectangular sea of uniform depth, when the angular speed of rotation is small. He also determined by an artifice, in the case of a square sea, the change in the longest free period due to rotation; his result was

\[ \sigma = \sigma_0 \pm 16 \omega^2 \]

\( \sigma_0 \) denoting the speed for no rotation. He verified his method of procedure by application to a circular sea of uniform depth.

T. Terada, on giving his form-corrections in connection with the determination of the period of the unidimensional seiche of a lake, gave also a correction to be applied to the observed length (1) to account for lateral motion of the water; this may be written

\[ \Delta l = \int \left( 1 - \cos \frac{nx}{l} \right) \left( \frac{dV}{dx} \right)^2 dx, \]

where \( x \) is measured along the medial line and \( b \) is the breadth of the lake corresponding to abscissa \( x \).

The free oscillations of water in non-rotating cylindrical basins, whose cross-sections are circular or nearly so, were examined by K. M. Ichi 5; he also determined the form of the section of a cylinder of given area for which the amplitude of the oscillation was a maximum.

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1 On Waves. Phil. Mag., 1, 257–79, 1876; Papers, 1, 251–71.
2 Theory of Sounds, II, 1878.
4 Manual of Tides, 1, 365–7; II, 335, 1897.
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theory, Harris concluded that, so far as semi-diurnal, diurnal and fortnightly forces are concerned, the earth behaves nearly as a rigid body.

A. Eneström has employed methods of the narrow-sea theory, including that of Chritsey, to determine the longitudinal seiches of enclosed seas and has applied his results to establish the existence of tidal motions in such basins. To account for the amplitude of the semi-diurnal tide at Chicago by the equilibrium theory he considered Lakes Michigan and Huron as a single body of water connected through Mackinaw Strait.

Rayleigh has given further details relative to the free oscillations of a "slowly rotating rectangular sea of uniform depth; the condition for slow rotation would be satisfied for a small sea in the neighbourhood of the equator.

In his "Théorie des Marées" H. Poincaré has transformed the dynamical equations of the tides from the differential to the integral form and suggested the use of the methods of Ritz and Fredholm for the determination of tides in restricted basins. Although the theory of integral equations has been considerably developed during recent years, the labour involved in obtaining particular solutions is prohibitive; the theory is, however, of value in establishing the existence of solutions.

R. Sternek applied the equilibrium theory to account for the spring tides of the Black Sea and obtained fair agreement with observation at Constanta, the only place for which values were then available. With the help of the storm-corrections he calculated the period of the free longitudinal oscillation of the basin and obtained the value 4.98 hours; he concluded that only forced semi-diurnal oscillations could exist in this basin.

The occurrence of a negative amphidromic point in connection with the semi-diurnal tides of the Black Sea was suggested by G. Wegener.

J. Proudman has examined certain cases of tidal motion in a "rotating sea of uniform depth. He approximated to both the free and forced tides of a nearly circular flat sea by a method similar to that used by Rayleigh in his "Theory of Sounds for a membrane; to the approximation considered he found the free periods to be the same as those for a circular sea of the same area. Considering

next the limiting forms of the forced tides as the period of the disturbance tends to become infinite, he found these forms to exist uniquely for flat closed seas of all shapes and to be always different from the equilibrium forms; he gave a complete expression for the limiting form in the case of a rectangular sea. He also approximated to the free oscillations of a small circular sea with its centre at a pole, taking account of the curvature of the sea. In a later paper Proudman further considered the limiting forms of long period 1913 A. (2) tides, discussing the existence, uniqueness and nature of the limiting forms for a general disturbing potential and the possibility of their application as approximations to actual tides; he treated as an example, among others, the case of a polar sea bounded by a parallel of latitude.

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F. Jager, following indications given by Poincaré, considered 1914 B. (1, 2) applications of the methods of Fredholm and Ritz to the problem of the tides in a basin with vertical boundaries; except in the case of resonance he established the existence of a unique continuous solution.

A hydrodynamical theory of the spring tides in the Black Sea has been given by Sternek. He calculated approximately the 1915 A. (2) forced longitudinal tides on the assumption that, so far as concerned these tides, the basin could be regarded as little different from a closed uniform canal; he first replaced the basin by a rectangular one of uniform cross-section having the same length and longest free period as the Black Sea, the free period being calculated according to the Japanese formulate, and then applied various corrections to the result. On allowing for the earth's rotation in accordance with the narrow-sea theory, he obtained a positive amphidromic point in the centre of the sea but, due to the crowding together of the cotidal lines, he concluded that the tidal motion was more nearly represented by a simple rocking about a.
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1 Manhattan of Tides, V, 446, 1908.

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central nodal line running from north to south. Observations, which were subsequently confirmed, indicated, however, the existence of a negative amphidromic point.

1919 A. (3) A. Defant used the step-by-step integration method to determine the free longitudinal period of oscillation of the Black Sea; he obtained the value 5.11 hours as compared with 4.99 hours obtained by Sterneck in 1913 by means of Morian's formula with the storm-corrections. 1920 B. G. I. Taylor applied his method for dealing with the reflection of a Kelvin wave at the closed end of a channel, to solve the problem of the free oscillations in a rotating rectangular basin of uniform depth. He showed that the oscillations may be divided into two classes, (a) those for which the elevation is symmetrical, (b) those for which it is anti-symmetrical with respect to the center, giving the details of the analysis in (b) but only the results in (a). He expressed the opinion that the free tides in a rotating rectangular basin consist of waves traveling round the basin in the direction of rotation and that the number of possible free periods is very much smaller than for a non-rotating rectangular basin, forming in fact a singly infinite series as compared with a doubly infinite series for a non-rotating basin. This latter is, however, at variance with Rayleigh's result for a slowly rotating rectangular basin (1903b, (1)).

1922 A. (1) The tidal motion of the Black Sea was again the subject of an examination by Sterneck. He replaced the basin by a rectangular one, having respectively a length and breadth approximately equal to the mean length and breadth of the sea and a uniform depth which gave the longest free period of the Black Sea; he then calculated the forced longitudinal and transverse oscillations on the hypothesis of no rotation and on these superposed oscillations due to the earth's rotation as required by the narrow-sea theory. Sterneck's calculated values for the spring tides compared well with observations, which existed for four stations, except that his amplitude for Odessa only amounted to about half the observed value; the phase distribution corresponded to a negative amphidromic point near the centre of the sea. For the diurnal tides a positive amphidromic point was obtained.

1923 B. L'Abbé C. Bertrand further developed the theory of integral equations in relation to tidal problems; the theory was first introduced by Poincaré.

The free oscillations of water in a non-rotating elliptical lake of uniform depth have been considered by H. Jeffreys. The slowest modes were calculated for ellipses of large and small eccentricities and the periods found to depend almost entirely on the major axis, in the higher modes nodal ellipses and hyperbolae may occur; he found that the longitudinal modes of a narrow lake are limiting cases of modes with no nodal ellipse. Jeffreys also questioned the validity of Taylor's statement in 1929, in connection with a rotating rectangular basin, that (1) there is only a single infinity of free modes of oscillation and (2) all free modes consist of waves moving round the basin in the direction of rotation.

In the fifth edition of his Hydrodynamics, Lamb gave an approximate method for calculating the speeds of the free oscillations of a rotating sea, based on the Calculus of Variations. He applied the method successfully to a circular basin of uniform depth; for a slowly rotating square basin, however, he found

\[ \sigma = \omega = \pm \sqrt{\sigma_0}, \]

which differs from the result obtained by Rayleigh in 1902.

J. W. Kurschakoff considered that the tides of the Black Sea could be explained as forced standing oscillations. Using the equation for damped forced oscillations

\[ \ddot{z} + 2\zeta \omega^2 z + \omega^2 z = \sin (\omega t - \gamma), \]

he applied the solution of this to determine the range of the spring tides atConstants; he obtained the value 7.7 cm., in agreement with observation.

Sterneck returned to the consideration of the tides of the 1925 A. (3) Black Sea after effecting the harmonic analysis for certain stations there; semi-diurnal constituents were known at five and the constituent K₁ at three stations. He noted that the results were in general agreement with those obtained from his theory of the tides of this basin given in 1922. He now gave, however, a more exact discussion, taking into account the exact form of the basin. He calculated the longitudinal oscillation, using the narrow-sea method of step-by-step integration, and then superposed on this two transverse oscillations, one given by the equilibrium theory, the other arising from the effect of the earth's rotation on the longitudinal oscillation. His theoretical results were in good agreement with the observed diurnal tide and he effected an improve-
central nodal line running from north to south. Observations, which were subsequently confirmed, indicated, however, the existence of a negative amphidromic point.

1918 A. (3) A. Defant used the step-by-step integration method to determine the free longitudinal period of oscillation of the Black Sea, he obtained the value 5.11 hours as compared with 4.98 hours obtained by Sterneck in 1913 by means of Morian's formula with the storm-corrections.

1920 B. G. I. Taylor applied his method for dealing with the reflection of a Kelvin wave at the closed end of a channel, to solve the problem of the free oscillations in a rotating rectangular basin of uniform depth. He showed that the oscillations may be divided into two classes, (a) those for which the elevation is symmetrical, (b) those for which it is anti-symmetrical with respect to the centre, giving the details of the analysis in (b) but only the results in (a). He expressed the opinion that the free tides in a rotating rectangular basin consist of waves travelling round the basin in the direction of rotation and that the number of possible free periods is very much smaller than for a non-rotating rectangular basin, forming in fact a singly infinite series as compared with a doubly infinite series for a non-rotating basin. This latter is, however, at variance with Rayleigh's result for a slowly rotating rectangular basin (1903B, (1)).

1922 A. (1) The tidal motion of the Black Sea was again the subject of examination by Sterneck. He replaced the basin by a rectangular one, having respectively a length and breadth approximately equal to the mean length and breadth of the sea and a uniform depth which gave the longest free period of the Black Sea; he then calculated the forced longitudinal and transverse oscillations on the hypothesis of no rotation and on these superposed oscillations due to the earth's rotation as required by the narrow-sea theory. Sterneck's calculated values for the spring tides compared well with observations, which existed for four stations, except that his amplitude for Odessa only amounted to about half the observed value; the phase distribution corresponded to a negative amphidromic point near the centre of the sea. For the diurnal tides a positive amphidromic point was obtained.

1923 B. L'Abbé G. Bertrand further developed the theory of integral equations in relation to tidal problems; the theory was first introduced by Poincaré.

The free oscillations of water in a non-rotating elliptical lake of uniform depth have been considered by H. Jeffreys. The slowest modes were calculated for ellipses of large and small eccentricities and the periods found to depend almost entirely on the major axis, in the higher modes nodal ellipses and hyperbolae may occur; he found that the longitudinal modes of a narrow lake are limiting cases of modes with no nodal ellipse. Jeffreys also questioned the validity of Taylor's statements in 1920, in connection with a rotating rectangular basin, that (1) there is only a single infinity of free modes of oscillation and (2) all free modes consist of waves moving round the basin in the direction of rotation.

In the fifth edition of his hydrodynamics, Lamb gave an approximate method for calculating the speeds of the free oscillations of a rotating sea, based on the Calculus of Variations. He applied the method successfully to a circular basin of uniform depth; for a slowly rotating square basin, however, he found

\[ s - a_i = \pm \frac{m \pi}{n^2}, \]

which differs from the result obtained by Rayleigh in 1902.

J. W. Curtiss, considered that the tides of the Black Sea could be explained as forced standing oscillations. Using the equation for damped forced oscillations

\[ z + 2k \zeta + m^2 \zeta = \sin (\sigma t - f), \]

he applied the solution of this to determine the range of the spring tides at Constanta; he obtained the value 7.7 cm., in agreement with observation.

Sterneck returned to the consideration of the tides of the Black Sea after effecting the harmonic analysis for certain stations there; semi-diurnal constituents were known at five and the constituent K_1 at three stations. He noted that the results were in general agreement with those obtained from his theory of the tides of this basin given in 1922. He now gave, however, a more exact discussion, taking into account the exact form of the basin. He calculated the longitudinal oscillation, using the narrow-sea method of step-by-step integration, and then superposed on this two transverse oscillations, one given by the equilibrium theory, the other arising from the effect of the earth's rotation on the longitudinal oscillation. His theoretical results were in good agreement with the observed diurnal tide and he effected an improve-
Proudman has given the solution of the free and forced tidal motion in a rotating semi-circular sea of uniform depth when the period of oscillation is half the period of rotation; the solution appeared in terms of infinite determinants and is applicable to semi-diurnal tides in a polar basin and to diurnal tides in latitude 30°. For a small deep basin of the dimensions of the Black Sea, it was noted that the first approximation to the forced tides was given by the equilibrium theory and the next approximation was examined.

The solution, involving Mathieu functions, of the free oscillations in a rotating elliptical lake of uniform depth was given by S. Goldstein. Later he obtained results for the free oscillations in a non-rotating elliptical canal, (an elliptical lake whose eccentricity is practically unity). He gave complete solutions for the first five normal modes of oscillation and simple approximate formulae for all modes; he found an increase in the speed of oscillation as compared with that in a rectangular canal of the same length and depth of approximately the same amount for all the modes; this he attributed to the narrowing of the ends. In a further paper Goldstein considered the free and forced tidal motion for an elliptical lake of uniform depth and also for one shallowing up to the coast, in the special case where the speed of rotation is twice that of the oscillation of the water. For the free oscillations in a basin for which the tide considered is of longest period and also for the forced tides, the free surface was found to be nearly plane, the cotidal lines were nearly straight and the corange lines nearly circular. The free wave moved round the basin in the direction of rotation. The amplitude of the forced tide was found to be greater than that given by the equilibrium theory and the phase-lag positive in the first and third and negative in the remaining quadrants.

The solution of the general problem of the free and forced tidal motion in a first rotating semi-circular sea of uniform depth has been given by Proudman; he had previously treated a special case. The general problem involved the solution of an infinite system of linear equations by the method of infinite determinants.

A first approximation for a deep sea of small horizontal extent was found to be given by the equilibrium theory and second approximations for both diurnal and semi-diurnal constituents were obtained. He illustrated his results numerically for a semi-circular sea of the same approximate dimensions and position as the Black Sea. He found a tidal wave progressing round the sea in the positive sense for diurnal and in the negative sense for semi-diurnal tides, as actually observed in the Black Sea, and his amplitudes were of the correct order of magnitude; the longest free period he computed to be 4.4 hours. He suggested that a complete discussion of tidal motion in the Black Sea would require an allowance for earth-tides. Later, in a note on forced tides in a lake, Proudman pointed out that, if the lake is deep and of small horizontal extent, a first approximation to the forced tides is furnished by the equilibrium theory, in spite of the possibility of free oscillations of the second class; the result is of importance in the determination of earth-tides. In a further note on the distribution of tides in a basin possessing a line of symmetry, which is a meridian or a parallel of latitude, he showed that the line of symmetry will be a cotidal line and that the cotidal and corange lines will be arranged symmetrically around this. When the equator is the line of symmetry there will be zero range along it for a diurnal and zero current for a semi-diurnal constituent.

The solution of the general problem of the free and forced tides in a rotating elliptical basin of uniform depth has been given with numerical details by Goldstein; the equation for the free periods appeared as the vanishing of an infinite determinant involving modified Mathieu functions. Free waves progressing round the basin in both positive and negative directions were found to be possible and an experimental test verified results obtained in connection with free oscillations of the basin. For the forced tides Goldstein illustrated his solution numerically by working out the case corresponding to that discussed by Proudman for a semicircle, having dimensions approximating to those of the Black Sea, and obtained very similar results. Thus a negative amphidromic point was derived for semi-diurnal and a positive one for diurnal tides, while the longest free period varied from 4.3 to 4.8 hours, accord-

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1 An erroneous statement, which occurs in Bulletin N. 15, is here corrected.
II. Historical Review
of Dynamical Explanations of the Tides of the Mediterranean, the Baltic Sea, the Gulf of Mexico and the Arctic Ocean.

By S. F. Grace
(University of Liverpool)

§ 1. This review has been compiled on similar lines to the preceding one and may, to a certain extent, be regarded as a sequel to it. For some purposes the basins of the Mediterranean, the Baltic Sea, the Gulf of Mexico and the Arctic Ocean may be treated as though they were non-elongated and closed, though the more recent explanations of the tides of these basins depend mainly on narrow-sea methods, as the subsequent discussion will show. The Mediterranean will frequently be considered in two parts, the west and east basins; the east basin is taken to mean the region east of the Strait of Tunis but not including the Adriatic and Aegean Seas; the Arctic Ocean is mainly restricted to mean the basin north of Greenland, the American Archipelago, Alaska, Siberia, Franz Josef Land and Spitzbergen.

In compiling this review and also the preceding one, the writer has been much indebted to Professor J. Proudman for permission to use his notes relative to many of the publications under discussion.

The Mediterranean Sea.

§ 2. D. Bernoulli, 1 comparing the range of the tide for an enclosed sea with that for an ocean covering the earth, indicated why the tides of the Mediterranean must be much smaller than those of the ocean.

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The view, which was originally held regarding the tides of the west basin of the Mediterranean, was represented by Hermann Berghaus in his *Physical Atlas*; these tides were not considered to be independent but due to a wave from the Atlantic Ocean progressing round the basin in a positive (counter-clockwise) sense.

R. A. Harris applied the equilibrium theory to determine the semi-diurnal tides of the East Mediterranean, which he considered as a closed sea; he obtained fair agreement with observation at the east and west ends as regards phase but the calculated amplitudes were too small.

G. H. Darwin expressed the opinion that the tides of the Mediterranean were but little influenced by the Strait of Gibraltar; he considered the Mediterranean to be virtually a closed lake but that its tides would more nearly resemble those of two lakes.

A rough explanation of the semi-diurnal tides of the Mediterranean has been given by Harris; he treated the east and west basins as closed and applied the equilibrium theory to each. He obtained a negative amphidromic point near the centre of each basin, but in the case of the west basin he stated that the tides would be influenced through the Strait of Gibraltar by those of the Atlantic Ocean. In connection with the diurnal tides of the whole basin he assumed that the Mediterranean may have a free period of oscillation approximating to 24 hours. Later he developed the theory of the amphidromic point and supplemented his explanation of the semi-diurnal tides. He constructed a chart of cotidal lines for the Mediterranean which depended rather on hypothesis than on an empirical basis. For the west basin he considered a wave from the Atlantic to be important; the cotidal lines were fairly close together in the neighbourhood of Cartagena but between the Balearic Islands and the Italian coast only one hour's tide-interval occurred. In the east basin the cotidal lines corresponded roughly to a negative amphidromic point centred near the west of Crete; the lines were close together to the north and south of the island and far apart to the west and east, but to arrive at this distribution Harris had to assume the phases in the Aegean Sea and along the North African coast, as no observations were available. The cotidal lines in the Strait of Tunis were crowded together.

L. de Marchi attempted an explanation of the semi-diurnal tidal motion of the Mediterranean, involving the idea of a bodily transference of water following the motion of the moon; he was unable to account in a satisfactory manner for the variation of phase over the sea.

From an examination of existing observations O. Kriemel suggested that the semi-diurnal tides of the west basin might be accounted for by means of a forced oscillation about a central nodal line, running approximately north-south through Minorca, with possible interference from the Atlantic.

R. Stornieck, studying the geographical distribution of the observed semi-diurnal tides at stations in the west basin, discovered the existence of a nodal line very much to the west of the centre of the basin. Relying on the opinion that the currents were negligible through the Strait of Gibraltar and between Sicily and Tunis, he assumed for this nodal line an oscillating region extending from Gibraltar to the Balearic Islands; the remaining portion of the Western Mediterranean he considered to constitute two further oscillating regions, one between the Balearic Islands and Sardinia and the other the Tyrrhenian Sea, in each of which he assumed a closed nodal line to exist. Between Sicily and Tunis he found a positive amphidromic point.

Observations made shortly afterwards by G. Grabhöwitz in the Tyrrhenian Sea indicated that a closed nodal line there did not exist; he also suggested that the excentrical nodal line in the west of the basin was somewhat curved.

This led Stornieck to abandon the idea of closed nodal lines and to investigate the flow of water into and out of the Western Mediterranean. He made a detailed examination of the current observations taken by G. S. Nares in 1871 in the Strait of Gibraltar and found them able partially to account for the position of the excentrical nodal line, the discrepancy he attributed to the interchange of water between the west and east basins of the Mediterranean. He then applied the equilibrium theory to both west and east basins, correcting the results in order to preserve the calculated interchanges of water through the connecting passages. With the help of the storm-corrections he calculated the periods of the free longitudinal oscillations of these basins and, from the fact that they were considerably less than 12 hours, concluded that only forced semi-diurnal oscillations could exist in the basins.

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Steneck later supplemented his theory on the tides of the Mediterranean, using the results of the harmonic analysis of observations at a number of stations there. He found that the separate semi-diurnal constituents conformed with his previous theory on the spring tides, the nodal lines of the longitudinal oscillations lying west of Algiers, across the Strait of Tunis and off the west end of Crete. He next surveyed the diurnal tides, assuming longitudinal motion unaffected by the earth’s rotation and also that the tide at a place consists of a co-oscillation of the whole basin with the Atlantic superimposed on a forced oscillation; the co-oscillation has the same (or opposite) phase as the diurnal tide at Cadiz and the forced tide the phase-lag 90° or 270° referred to the mean meridian of the basin (15° E. of Greenwich). For the co-oscillation Steneck found two nodal lines, one near Gibraltar and the other across the Strait of Tunis; the forced tide had nodal lines at the mouth and in the Ionian Sea. He further found that the forced tide was in good agreement with the diurnal tide determined by the equilibrium theory. He reproduced his discussion on the diurnal tides in a subsequent paper.

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The most satisfactory discussion relative to the tides of the Mediterranean seems to be that given by Steneck in 1915 and it is to be regretted that he has omitted to furnish a cotidal chart for the region, but he contented himself with comparing his theoretical results with then-existing observations. It is, however, an open question at present, how far it is legitimate to determine the tidal motion of a non-elongated sea by applying narrow-sea methods in two or more directions and then combining the results, although it appears, from the fair agreement between the calculated values for the Mediterranean and those derived from observation, that the error involved will be small.

The Baltic Sea.

§ 3. Krümmel has commented on the marked difference in character between the tides of the North Sea and the small tides of the Baltic Sea; whereas in the North Sea the semi-diurnal tide predominates, on proceeding from the North Sea into the Baltic Sea tides of mixed type are first encountered but in the innermost regions, the Gulf of Bothnia and Finland, the diurnal tide is the more important. Krümmel attributed to resonance the phenomenon of the predominating diurnal tide; for the Gulf of Bothnia he obtained a free period of oscillation of 24.1 hours, using a convex parabolic approximation to the normal curves for the basin, while for the Gulf of Finland, using Moritz’s formula with smooth corrections, he found a free period of 24.4 hours.

A monograph on the tides of the Baltic Sea (including the Gulf of Finland) was written by F. Witting. He calculated the free periods of the longitudinal oscillations and also of the transverse oscillations at various sections, but found serious discrepancies between the results given by the storm corrections of Iisani and Terada and those given by Chrystal’s method; he de-
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\(^1\) Sur les marées de la Méditerranée orientale, Comptes Rendus, 191, 275, 1923.
cluded in favour of the latter but modified his results to allow for friction. He then considered a closed rectangular basin of uniform depth having the same length and longest free period as that of the actual sea and, adding to the equation of motion given by the canal theory a term proportional to the current to represent the effect of friction, he solved this equation both for free and forced tides. Witting next compared the results of the harmonic analysis of observations with those derivable from theory. He explained in a general manner the diurnal tides as consisting of an inverted forced unimodal longitudinal oscillation on which was superposed a direct forced unimodal transverse oscillation; the semi-diurnal tides he considered to consist of a forced trimodal longitudinal oscillation together with a direct forced unimodal transverse oscillation and a longitudinal trimodal co-oscillation with the tidal motion at the mouth. The discussion omitted from consideration the Guls of Bothnia and Riga; the tides here, according to Witting, are more or less independent.

The assumed semi-diurnal oscillations of the Baltic Sea involve three amphidromic points, the two outer ones being positive and the inner one negative; observation appears to give much evidence for the existence of the two positive amphidromic points but very little for that of the negative one as Witting also points out. In this connection it should be noted that his use of a rectangular basin prohibits the possibility of a forced binodal longitudinal oscillation but it is conceivable that for the actual basin the oscillation may be binodal. If this were so then along its two nodes the transverse oscillations would have to be in opposite phases to make both amphidromic points have the same sign. Such transverse gradients are produced by the earth's rotation, full allowance for which may not have been made by Witting.

The question of tidal friction in seas has been examined by H. Joffreux from observations of tidal currents, on the assumption that the force of friction at the bottom is given by \( \alpha \left( \frac{\partial v}{\partial t} + \mathbf{u} \times \mathbf{v} \right) \) per unit area, where \( \mathbf{v} \) is the velocity of slip and \( \mathbf{u} \) the density. He formed the opinion that in basins like the Mediterranean and the Baltic Seas there is little or no dissipation of tidal energy, such basins being practically land-locked and the tidal motion there so small.

§ 4. It was noted by W. Ferrel that in the Gulf of Mexico the diurnal tide predominates; he suggested that the tidal motion of this basin was dependent partly on a progressive wave from the Atlantic Ocean and partly on the tides generated in the basin itself. Harris (loc. cit.) considered that the Gulf of Mexico could be regarded as closed so far as concerned the semi-diurnal tides but that the diurnal tides would be influenced through Yucatan Channel by those of the Atlantic. Later he considered the semi-diurnal tides to consist of an equilibrium tide on which was superposed a progressive wave through the Florida Channel; as regards the diurnal tides he stated that the Gulf of Mexico and the Caribbean Sea would form an oscillating region with a nodal line extending from the west of Haiti to Nicaragua, thus making the tides simultaneous in the Gulf. In a further contribution he supplemented his explanation of the semi-diurnal tides; he constructed a cotidal chart for the region and decided that the cotidal lines would not be amphidromic, on account of a progressive wave from the Florida Channel, diminishing westwards.

The predominating diurnal tidal motion of the Gulf of Mexico was similarly supposed by A. Endröis to be due to resonance effects. By considering the Gulf of Mexico and the Caribbean Sea as a single oscillating body of water with a node at the Lesser Antilles he obtained a period near to 24 hours, using a convex parabolic approximation to the normal curve for the basin.

G. Wegener has suggested that resonance in the Gulf of Mexico is probably due to the fact that the west-east oscillation has a free period of 24.8 hours; he considered the basin as open in the east and applied a smooth-correction.

Storzreck, discussing the tides of the oceans, considered them to consist of two superposed standing oscillations differing in phase by \( \pi/2 \) and evaluated the amplitudes from existing observations. On applying the method to the semi-diurnal tides of the Gulf of Mexico he obtained cotidal lines for the region, which corresponded to a negative amphidromic point near the centre of the basin; he accounted for the diurnal tidal motion by means of a simple co-oscillation with the Atlantic, having opposite phase.

From the point of view of the equilibrium theory and the fact that the diurnal tide is dependent on the declination of the
The Gulf of Mexico.

§ 4. It was noted by W. Ferrel that in the Gulf of Mexico the diurnal tide predominates; he suggested that the tidal motion of this basin was dependent partly on a progressive wave from the Atlantic Ocean and partly on the tides generated in the basin itself.

Harris (loc. cit.) considered that the Gulf of Mexico could be regarded as closed so far as concerned the semi-diurnal tides but that the diurnal tides would be influenced through Yucatan Channel by those of the Atlantic. Later he considered the semi-diurnal tides to consist of an equilibrium tide on which was superposed a progressive wave through the Florida Channel; as regards the diurnal tides he stated that the Gulf of Mexico and the Caribbean Sea would form an oscillating region with a nodal line extending from the west of Haiti to Nicaragua, thus making the tides simultaneous in the Gulf. In a further contribution he supplemented his explanation of the semi-diurnal tides; he constructed a cotidal chart for the region and decided that the cotidal lines would not be amphidromic, on account of a progressive wave from the Florida Channel, diminishing westwards.

The predominating diurnal tidal motion of the Gulf of Mexico was similarly supposed by A. Enríquez to be due to resonance effects. By considering the Gulf of Mexico and the Caribbean Sea as a single oscillating body of water with a node at the Lesser Antilles he obtained a period near to 24 hours, using a convex parabolic approximation to the normal curve for the basin.

Wegenmann has suggested that resonance in the Gulf of Mexico is probably due to the fact that the west-east oscillation has a free period of 24.8 hours; he considered the basin as open in the east and applied a smooth correction.

Størmer, discussing the tides of the oceans, considered them to consist of two superposed standing oscillations differing in phase by π/2 and evaluated the amplitudes from existing observations. On applying the method to the semi-diurnal tides of the Gulf of Mexico he obtained cotidal lines for the region, which corresponded to a negative amphidromic point near the centre of the basin; he accounted for the diurnal tidal motion by means of a simple co-oscillation with the Atlantic, having opposite phase.

From the point of view of the equilibrium theory and the fact that the diurnal tide is dependent on the declination of the
The Arctic Ocean

§ 5. In connection with the spring tides of the Arctic Ocean, W. Whewell suggested that a progressive wave would enter the basin between Greenland and Norway, traverse the Polar basin and end its course on the shores in the neighbourhood of Bering Strait.

An explanation of existing observations relative to the semi-diurnal tides of the Arctic Ocean has been given by Harris. He mentioned that the semi-diurnal equilibrium tide in this basin would be very small since the corresponding forces vanish at the Pole and, on account of the dimensions of the basin, there would be no resonance; practically all the tidal motion would be derived from the Atlantic, mainly through the Greenland Sea. He considered a wave through the Greenland Sea to divide into two branches. One branch progressed westward north of Greenland and the American Archipelago, the other branch progressed eastward with diminishing range north of Franz Josef Land, Siberia and Alaska and disappeared in Coronation Gulf; the tides of the Kara and White Seas were supposed to be due to a branch of the latter wave turning around Franz Josef Land and entering these...

1923 A. moon, H. Peters attempted to establish the diurnal character of the tides of the Gulf of Mexico and of the South China Sea.

1923 A. Relative to the tides of the Gulf of Mexico, A. Defant indicated the probable existence of a co-oscillation with the tidal motion outside either the Florida or Yucatan Channels. This he suggested, together with a forced oscillation of the basin regarded as closed, might account for the semi-diurnal tidal motion; on account of the earth's rotation there would be an amphidromic point. The diurnal tides, he considered, would be mainly co-oscillation tides.

The Gulf of Mexico appears to be a suitable basin for a detailed examination of its tides by a method similar to that used by Sternbeck for the west basin of the Mediterranean, but so far no such detailed examination has been made. As Defant has suggested the tides could probably be accounted for by forced tides generated in the basin itself, together with co-oscillations with the tidal motion outside the Florida and Yucatan Channels.

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It was suggested by H. Poincaré that the region of the Arctic Ocean, which Harris assumed to be traversed by a wave from the Greenland Sea, would constitute an oscillating region in which resonance with the semi-diurnal disturbing forces may occur; since the earth's rotation has its maximum effect in the neighbourhood of the Pole this would probably be an amphidromic region, as indicated by Harris' cotidal lines. He considered Harris' assumption of three progressive waves entering the region of the Arctic Ocean to be impossible, since this would necessitate disturbing forces in the region of considerable amount whereas they are practically negligible.

On comparing the harmonic constants for the Arctic and North-east Atlantic tides, Krümmel decided that, on the whole, there was a definite connection between these tides.

Harris, after collecting all available information relative to the tides of the Arctic Ocean, noted that observations indicated that the diurnal tides in certain parts of the basin are largely equilibrium tides; in other parts they are propagated from the Atlantic, though the mode of propagation was not made clear. In connection with the semi-diurnal tides he now dispersed with a branch of the wave from the Greenland Sea progressing westward and obtained a small amphidromic region north of Greenland.

From observations made in Beaufort Sea, W. Bell Dawson concluded that the tidal wave enters this sea from the north and progresses southward and then westward, in accordance with the view that the main semi-diurnal tide of the Arctic Ocean enters from the North Atlantic between Greenland and Norway.

J. E. Fjeldstad examined the semi-diurnal tides of the Arctic with the help of new data from the 1922-1923 expedition; he constructed a cotidal chart for the greater part of the basin but omitted from consideration that portion which lies immediately north of the American Archipelago. The chart was somewhat similar to that of Harris in the region of east longitude but differed in the region of west longitude, mainly in that the cotidal lines were
here less crowded together; he gave reasons for the non-existence of the tract of land assumed by Harris and favored the hypothesis of F. Nansen of a deep polar basin. He explained the tidal motion by means of a progressive wave from the Atlantic of diminishing range, which entered between Greenland and Spitzbergen and was propagated with a speed dependent on the depth, rapidly at first across the deep polar basin and then more slowly on approaching Bering Strait; the time to traverse the basin from Greenland Sea to Wrangel Island he found to be about 12 hours, whereas Harris' chart suggested about 18 hours. He also found that the tides on the North Siberian Shelf were considerably influenced by friction, one of the principal causes of which he attributed to the ice-sheet.

Defant included the Arctic Ocean in a hydrodynamical explanation of the tides of the Atlantic; he considered the two basins as a canal open at the south end but virtually closed across Bering Strait. So far as concerned the semi-diurnal tidal motion of the Arctic he considered this to consist of a longitudinal oscillation, the resultant of a co-oscillation and a forced oscillation determined by narrow-sea methods of step-by-step integration, on which was superposed a transverse oscillation due to the rotation of the earth. He obtained two positive amphidromic points, one centered near Spitzbergen and the other on the opposite side of the Pole, which he stated were in agreement with observation. For the diurnal tides he obtained a positive amphidromic point south of Spitzbergen but in the greater portion of the basin the tides were accounted for by a progressive wave traversing the basin from the Greenland Sea to Bering Strait in about 12 hours. He noted that his distribution of cotidal lines was in accordance with observation and that the amplitudes were of the correct order of magnitude.

An explanation of observations relating to the spring tides on the North Siberian Shelf has been furnished by H. Sverdrup. He found the cotidal lines to have a marked tendency to run parallel to the isobaths; their course differed somewhat from that given by Harris, partly on account of his assumption of a tract of land in the Arctic, but more nearly resembled that by Hjelstadb. Sverdrup concluded that tidal phenomena on the Shelf do not indicate the existence of land within the unexplored region in the neighbourhood of the Pole.

With regard to Defant's assumption that the Atlantic and Arctic Oceans can be considered as a single canal, closed across Bering Strait and for which narrow-sea methods are applicable to determine the longitudinal oscillations, it was suggested by Sternbeck that this assumption is untenable. He pointed out that, at least for the semi-diurnal tide, this would involve an oscillation of considerable magnitude at the closed end, whereas observations north of Alaska and East Siberia indicate that both the semi-diurnal and diurnal tides have here remarkably small amplitudes. With the cooperation of J. Foll, Sternbeck subsequently gave a theoretical discussion relating to the M2 and K1 constituents of tidal motion in the Arctic Ocean. He considered the basin to consist of a very deep inner region together with a shallow boundary region and calculated the forced tides of the deep central region, on the assumption that the basin was closed, as far as possible in accordance with the method he employed for the Black Sea (1929 A. (8)), but in the case of the M2 tide he found it necessary to consider also a co-oscillation with the Atlantic. He found the M2 tide to depend mainly on this co-oscillation but the K1 tide to be a forced tide; on account of transverse oscillations he obtained amphidromic regions, a positive one for M2 centered near the Pole and a negative one for K1. He then considered that the tidal waves from the deep central region would be propagated over the shelf region as progressive waves; in the latter region, therefore, dissipation of tidal energy would occur. A cotidal chart for the M2 tide was constructed, which agreed well with observation. The deep central region, which included depths greater than 200 m., Sternbeck computed to contain 97% of the water in the whole basin. He stated that it would be erroneous to apply narrow-sea methods to determine the longitudinal oscillations of the deep basin together with the shelf region, as Defant had done, since the velocity across a section would not be uniform.

Defant has suggested that friction may be important in the Arctic Ocean, on account of the existence of the ice-sheet and also of the shallow shelf region, and that for this reason the calculations of the tides of this basin already effected may be somewhat illusory; in particular the co-oscillation of the deep polar basin with the Atlantic may not have the character of a simple standing wave as Sternbeck assumed. He pointed out that all existing observations refer to the shelf region and suggested that until

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observations were available for the deep central basin further calculations relative to the Arctic tides may be unprofitable. From the morphological relations of the basin and the condition of its surface he expressed the opinion that the tidal motion has rather the character of a progressive wave from the Atlantic.

Erected in Bulletin N.15.

Page 4, line 28, for \( \epsilon \) read \( \epsilon' \).
- 6, for \( \epsilon \), read \( \epsilon' \).
- 10, for \( K_2 \) read \( K_2 \).
- 13, for 1916 A. read 1956 A.
- 15, for \( \xi \) read \( \xi \).
- 18, for \( \xi \) read \( \xi' \).
- 18, for \( \xi' \) read \( \xi' \).