Historical Review of Dynamical Explanations of Tides and Seiches in Narrow Seas and Lakes

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1.

The scope of the following review is determined by mathematical considerations. From a geophysical point of view the motions considered are of two distinct types, viz. (1) the harmonic tidal constituents corresponding directly to the generating forces of the moon and sun, and (2) the free oscillations of lakes. The mathematical methods used for their investigation, however, are similar.

A narrow sea or lake is taken to be one in which, for the motions considered, the currents are mainly in the direction of the medial line of the basin.

In addition to publications (1) relating to actual seas reference is also made to those relating to geometrically simple basins. Publications dealing with the results of the analysis of observations

(1) Most of the publications referred to are enumerated in the Tidal Bibliography published in these Bulletins. Reference will be made by means of the date and the group-letter used in the Bibliography. When in any group several publications by the same author have the same date, a distinguishing number will be added to the reference. Thus 1914 A (1) opposite the name Defant, refers to the first publication of Defant given under the year 1914 in group A of the Bibliography.

Such references placed in the margin serve as a chronological table; further, it should be noted that the letters A and B correspond respectively to actual seas and to geometrically simple basins as the principal subject matter.
are only mentioned when they have a direct bearing on dynamical investigations. Motions depending primarily on variations of density, e. g. submarine tides and temperature-seiches, are omitted from consideration.

As the chief object of the review is to give an account of recent research, later publications are treated in greater detail than earlier ones, but those issued after 1928 are not included.

2. Notation.

The following symbols are used:

- \( l \) = the length of the medial line of the basin,
- \( b \) = the breadth of the basin,
- \( h \) = the depth of water below any point of the mean surface,
- \( A \) = the area of a vertical section transverse to the medial line,
- \( x, y \) = the coordinates of a point, \( x \) measured along and \( y \) transversely to the medial line,
- \( u, v \) = the components of current in the directions of increasing \( x, y \),
- \( \theta \) = the colatitude,
- \( \Omega \) = the angular speed of the earth's rotation,
- \( \omega \) = \( \Omega \cos \theta \),
- \( g \) = the acceleration of gravity,
- \( \xi \) = the horizontal displacement of a water-particle in the direction of increasing \( x \),
- \( \zeta \) = the elevation of the surface relative to the bottom,
- \( \zeta_0 \) = the elevation of the bottom,
- \( -g \xi \) = the potential of the astronomical tide-generating forces,
- \( -g \zeta_0 \) = the change in potential due to both earth-tide and water-tide,
- \( \Pi \) = the amplitude of a wave,
- \( T \) = the period of a harmonic oscillation,
- \( \sigma \) = \( 2 \pi / T \),
- \( \beta \) = a constant depending on frictional influences,
- \( t \) = the time.

The dynamical theory of a free tidal wave progressing along a canal of uniform rectangular section, when the effect of the earth's rotation is neglected, was first given by Lagrange.

The dynamical theory of the small motion of a liquid in a closed basin of finite depth was considered by J. R. Merian, continuing the work of Poisson and Cauchy who had restricted themselves to the case where the liquid was unbounded and of infinite depth. He gave the theory of the free longitudinal oscillations in a non-rotating closed rectangular basin of uniform depth, but there is no evidence that it was immediately applied to explain such phenomena as seiches in actual lakes, though these had been observed and described in the previous century. His formula for the longest free period,

\[ T = \frac{2 \pi}{\sqrt{g h}} \]

will be referred to as "Merian's formula."

The case of progressive waves in a long stationary canal of rectangular section, the breadth and depth of which varied gradually, was investigated by G. Green; he found the speed of propagation to be \( \sqrt{g h} \) and the elevation of the wave to be inversely proportional to \( \sqrt{\ell} \).

In his "Tides and Waves," G. B. Airy greatly advanced the theory of tidal motion in canals, with applications to rivers and narrow seas. In addition to the closed lake treated by Merian, he considered a gulf open at one end and a channel open at both ends; he investigated the effects of varying section and of friction proportional to the velocity, but entirely neglected the rotation of the earth. On the semidiurnal tides of the Adriatic and the Bay of Fundy he offered a general explanation of such observations as were available, by assuming a longitudinal standing oscillation with a node near the mouth; on the observed tides of the English Channel he referred to the theory of the superposition of two progressive waves travelling in opposite directions but was unable to account for the transverse gradients of the Irish Sea. A little later Airy (1) returned to the explanation of the semi-diurnal tides of the Irish Sea.

Sea with the additional help of an extensive series of observations made by the Ordnance Survey on the coasts of Ireland. So far as concerned these observations he accounted for the semi-diurnal tides of the Irish Sea by a large standing wave, having a node near Courtown, on which was superposed a small progressive wave; the North Channel he regarded as a canal joining two tidal seas.

1879 B.

Merian’s formula was applied by F. A. Forel, who had made extensive observations on the seiches of Lake Geneva and other Swiss lakes. He found that the formula gave a result agreeing closely with his observations of the period of the slowest seiche.

It was Sir W. Thomson (afterwards Lord Kelvin) who introduced the rotation of the earth into the tidal dynamics of small seas and thus accounted for the transverse gradient across a narrow sea. He gave the theory of a free wave progressing along an open rotating channel of uniform rectangular section in which the currents are entirely longitudinal. Such a motion, defined by

$$\xi = H e^{-\frac{1}{2} \omega t} \cos \omega \left| t \pm x/V(gh) \right|,$$

will for convenience be referred to as a "Kelvin-wave." By two superimposed Kelvin-waves travelling in opposite directions he accounted in a general manner for some of the observed features of the tides of the English Channel and Irish Sea.

It will be seen below that it was not until 1918 that the principles laid down by Airy and Thomson were fully used to give detailed explanations of observed tides in narrow seas.

The first serious attempt to submit seiche-phenomena to detailed calculations was made by P. du Bois. In connection with the longitudinal seiches in lakes, he suggested that variable depth could be allowed for in Merian’s formula by replacing \( l/V \) by a mean value taken over the lake. His formula, which may be written

$$T = 2 \int_0^l dx/V(gh),$$

proved to be an important advance.

1894 A.

C. Börgen offered an explanation of the semi-diurnal tides of the Irish Sea by means of two progressive waves travelling in opposite directions, whose amplitudes were progressively reduced by friction. The transverse gradients he tried to explain without the action of the earth’s rotation.
In his "Hydrodynamics," H. Lamb gave the solution of the equations of longitudinal oscillations for certain simple basins; in these the sections were rectangular and the breadth and depth followed simple laws of variation from one end to the other.

An explanation of the semi-diurnal tides of the English Channel and the Flemish Bight was offered by Börger on similar principles to those which he had given of the Irish Sea tides. He returned to the subject in a supplementary paper. (1808 B)

In his "Manual of Tides," R. A. Harris attempted in a rough way to explain the semi-diurnal tides of the Red Sea as consisting of a standing wave, maintained by the local astronomical forces, on which was superposed a progressive wave from the Indian Ocean; the semi-diurnal tides of the Guls of Suez and Akaba he accounted for as being mainly co-oscillations with the Red Sea. In the latter year he explained the nature of the "narrow sea type," of amphidromic point and offered rough explanations of the semi-diurnal tides of the English Channel, the Irish Sea, the Adriatic and the Persian Gulf. The tides of each of the first three he considered as consisting partly of standing waves and partly of progressive waves from the larger connected bodies of water; the tides of the Persian Gulf he considered as a simple progressive wave from the Indian Ocean, somewhat retarded by friction. Only in the case of the North Channel of the Irish Sea did he obtain an amphidromic point of the narrow sea type.

4. THE NARROW SEA THEORY.

A new departure was made by G. Chrystal in his mathematical work on the longitudinal seiches of lakes. Starting with the equations

\[
\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{a} \frac{\partial^2 \xi}{\partial x^2}, \quad \frac{\partial}{\partial x} (\Lambda \xi) + b \xi = 0
\]

(1), (2),

as used by Green (1887 B), he introduced a function, the graph of which he called the "normal curve" of the lake. Writing

\[ p = \int a \, b \, dx, \quad q = \int b \, dx, \]

a point on the normal curve has abscissa \( q \) and ordinate \( p \). Chrystal then developed the solution of his differential equations for
special cases similar to those considered by Lamb, with the object of replacing the normal curve of an actual lake by a small number of algebraically simple curves. This return to the fundamental differential equations became very important when it was followed by investigators of tides; the explanations in terms of simple progressive waves had figured too largely and even when legitimate they were often merely kinematical and not dynamical. Later, with the help of E. M. Wedderburn, he applied his theory to Lochs Earn and Treig, replacing the normal curves by portions of parabolas, and obtained very good agreement with the observed values of the periods and the positions of the nodes.

Following the indications of Chrystal, D. Isitani and T. Terada showed how the variation of section in a lake could be allowed for in Mariam's formula, provided that the shape of the lake did not differ considerably from that of a rectangular tank; the theory of the method is similar to that of air-vibrations in a pipe of variable section given by Rayleigh. The result may be written

\[ T = \frac{2I}{V(gh)} + \frac{l}{V(gh)} \left( \frac{b}{b_o} + \frac{A}{A_o} - 2 \right) \cos \frac{2\pi x}{l} dx \]

where \( b_o \) denotes the mean breadth, and \( A_o \), the mean area of cross-section. The second term in the expression for \( T \) will be referred to as the "form-correction." The formula was tested by Terada by application to Lake Hakone. In the following year K. Honda, T. Terada, Y. Yoshida and D. Isitani extended the results of Isitani and Terada to the case of bays open to the sea, by the introduction of an "end-correction," based on Rayleigh's discussion of the two-dimensional air-vibrations in an open pipe. They considered various bays on the coasts of Japan and elsewhere; the free period of the Adriatic they computed to be about 15 hours.

R. v. Sternneck (sen.) showed that observations on the semi-diurnal tides of the Adriatic indicated the existence of an amphidromic point about one quarter to one third of the way along the sea from the closed end; the cause of this he attributed to the superposition of two oscillations, approximately at right angles and with a phase-difference of \( \pi/3 \).

In his "Théorie des Marées," H. Poincaré questioned Harris' explanation of the semi-diurnal tides of the Persian Gulf; he also noted the existence of an amphidromic point of the narrow sea type in the Adriatic.
In connection with the slowest seiche of Lake Hakone, S. Nakamura and K. Honda compared the various methods of calculating the period.

In his 'Handbuch der Ozeanographie', O. Krommel applied simple calculations of the Merian-type to explain roughly the tides of the Bay of Fundy and of the Gulf of California.

The slowest free co-oscillation of the Adriatic with the Mediterranean was calculated by A. Defant. He used the formulae of the Japanese investigators, applying both form and end-corrections and obtained the value 22.4 hours for the longest period. According to R. Witting, however, Defant's computation is in error and should have led to the value 15.8 hours. R. Sterneck later (1915 A (1)), obtained from independent measurements a value of about 16 hours, in agreement with Witting's result; he suggested that the Japanese formulae may give wholly incorrect results when the form-correction is large.

A. Blondel gave a dynamical discussion of the constituents $M_2, S_2, N_2, K_1, O_1, P_1$ of the Red Sea, using the actual dimensions of the basin. He applied dynamics by means of the Calculus of Variations and used the method of Ritz indicated by Poincaré, but he obtained no agreement between theory and observation. He attributed his failure to his neglect of friction; Poincaré had suggested, however, the possibility of an earth-tide.

The semi-diurnal tides of the Mediterranean including, in particular, those of the Adriatic, were studied by Sterneck. He calculated the current through the Strait of Otranto from observations made in the Adriatic; he also calculated, with the help of the form-correction, the position of the nodal line in the free longitudinal co-oscillation with the Mediterranean, and this he found to lie about 30 km. S. E. of the position of the established amphidromic point. In the same year W. v. Kesslitz discussed the tides of the Adriatic on the results of harmonic analysis at stations on the east coast. He considered the constituents $M_2, S_2, N_2, K_2, K_1, O_1, P_1$, but many of the constants were wanting. By 1914 Sterneck had recognised the semi-diurnal amphidromic point of the Adriatic to be of the narrow sea type. Calculating the longitudinal currents from the observed elevations by means of the equation of continuity, he obtained the transverse gradients all along the sea from the equation:

$$\frac{\partial \xi}{\partial y} = - \frac{2 \Omega u \cos \theta}{y},$$

(3)
His determination of the effect of the earth's rotation compared favourably with results which he derived directly from the observations. In the same year Defant showed that there was very good detailed agreement between the observed tides of the Adriatic, both for semi-diurnal spring tides and the constituent K_2, and the theoretical results for a basin of rectangular section and uniform breadth whose depth varied as the distance from one end. He took the length as that of the sea but his choice of breadth and depth appeared to be somewhat arbitrary; the depth of the Adriatic, in particular, diverges considerably from the assumed values, especially in the southerly half of the sea.

J. H. M. Wedderburn considered certain particular cases of the propagation of tidal waves in channels of rectangular section, whose depth followed simple laws of variation. A direct mathematical process for integrating Chrsyal's differential equation without approximating to the normal curve was given by J. Proudman. He obtained a general equation for the free periods and a general expression, in terms of the period, for the motion in any free mode, as well as a complete expression for the forced motion.

In the following year Sternshek also abandoned the use of results from algebraically simple basins by inserting the actual dimensions of the basin of the Adriatic into the fundamental differential equations (1) and (2). He did not, however, give a complete step-by-step integration of these equations, but used the observed elevations for the semi-diurnal spring tides and the constituent K_2, and showed that by slight modifications they could be made to satisfy equation (2). From equation (1) he calculated the values of the corresponding periods and obtained close agreement with actual.

Later in the same year Sternshek applied step-by-step integration to the co-oscillations of various parts of the Mediterranean.

In the following year he gave an explanation of the peculiarities of the currents in the Strait of Choles produced by the difference in level of the water to the north and south; he assumed that the narrow Strait of Choles acted as a virtual barrier for both the tidal oscillations and the seiches of the northerly and southerly channels.

The step-by-step integration method used by Sternshek for the co-oscillation of a bay with the main body of water, formed the basis of a method exhibited by Defant for the determination of
seiche-periods; starting with different assumed values for the period, successive integrations are effected until the end conditions can be satisfied by interpolation. He applied the method to the Black Sea and to Lake Garda. Defant also gave detailed hydrodynamical explanations of the tides of the Red Sea and the Gulfs of Suez and Akaba and also of the Persian Gulf, both for semi-diurnal spring tides and for diurnal tides, by means of complete step-by-step integration. He found that in the spring tides of the Red Sea, the forced oscillation produced by the local astronomical forces is important as well as the co-oscillation with the Gulf of Aden; for the semi-diurnal tides of the Persian Gulf he obtained two amphidromic points. He further applied similar methods to the English Channel and the Flemish Bight but found it necessary to take account of the force of friction; this he assumed to be proportional to the current, so that instead of equation (1), he used

$$\frac{\partial^2 \xi}{\partial t^2} - \beta \frac{\partial \xi}{\partial t} - g \frac{\partial \xi}{\partial x}$$  \hspace{1cm} (4)

The forced oscillation produced by the local astronomical forces he found to be negligible. He considered separately the Gulf of St. Malo, the Thames Estuary and the Wash, finding the co-oscillations of these bays with the main body of water.

Kesslitz published a set of harmonic constants for the east coast of the Adriatic, which was more complete than those previously given, and then Sterneck gave a detailed hydrodynamical explanation of each of the constituents $M_2$, $S_2$, $N_2$, $K_2$, $K_1$, $O_1$, $P_1$ of the tides of this sea, friction being found unimportant. Sterneck also determined the period of the seiche which has a node in the Strait of Otranto and, after making the end-correction, obtained the value 23.34 hours; he concluded that the form-correction, which gave 16.4 hours, was here inapplicable. He introduced an inertia-correction to the equation (3) depending on the ratio of the period of a free transverse oscillation to the period of the actual motion.

Proudman's method (1914 B (2)) was applied by A. T. Dodson, R. M. Carey and R. Baldwin to the theoretical determination of the longitudinal seiches of Lake Geneva, which possesses a normal curve for which Chrystal's methods are quite unsuitable. The three slowest seiches were considered; the results for the unimodal and binodal oscillations were found to be in close agreement with observation, but no comparison was possible for the trinodal oscill-
lation. This method of determining free oscillations gives more mathematical satisfaction than any of the others but equal accuracy with less labour may probably be obtained by the tentative method of Defant. (1918 A (3)).

1920 A. (1). Defant applied to the Irish Sea methods similar to those which he had used for the English Channel; he considered separately the co-oscillations of the Bristol Channel, Liverpool Bay and Solway Firth with the main basin. In the latter cases he assumed the effect of friction to increase as the depth decreased.

5. EXTENSIONS OF THE NARROW SEA THEORY.

In the narrow sea theory the only condition required to be satisfied at a closed end is that the total flux of water shall be zero. The degree of error involved in this was investigated by G. I. Taylor. He considered the reflection of a Kelvin-wave at the closed end of a long rotating channel of uniform rectangular section. For the dimensions of the Irish Sea he found that the maximum transverse current at the closed end would be 0.5 knot, but that at the place of maximum longitudinal current it would only be $5 	imes 10^{-8}$ of the longitudinal current. Taylor also compared the semi-diurnal tides of the Bristol Channel with the frictionless oscillations of a basin of rectangular section, whose breadth and depth are both proportional to the distance from one end. For the distribution of range very good agreement was found, but the neglect of friction makes the tides simultaneous and leaves unexplained the two hour's time-difference along the channel. Taylor's solution for a channel increasing uniformly in breadth and depth was obtained later in a slightly different form by G. Greenhill.

1922 A. The laws of friction hitherto used in the subject under review have much to be desired. Defant showed, however, in his discussion on the Irish Sea, that the numerical value which he had assumed for the quantity $\beta$ in (4) led to a loss of energy by friction in agreement with a determination made by Taylor. (1918 A).

1922 A. (2). Sterneck gave a criterion for the separation of the forced oscillation from the co-oscillation with a connected body of water, different from that used by Defant. It consists in decomposing the elevations at a place uninfluenced by transverse motion, into two parts, one having the phase of the forced tide and the other the phase of the oscillation at the mouth. This he applied to the diurnal tides of the Mediterranean.

1923 F.
The free longitudinal oscillations of a narrow lake of uniform depth with an elliptic plan were considered by H. Jeffreys. Among other things he found that, to a first approximation, the more rapid periods are the same as for a lake of uniform breadth having the same length and depth.

N. Sen discussed certain types of solutions of the equations (1) and (2) corresponding to the motion of progressive waves, together with the laws of variation of breadth and average depth at a section of the canal for which such types of motion could exist. His examination was similar to that made by Wedderburn (1914 B) for variation of depth alone.

L. Matteuzzi gave a method for the determination of the free and forced seiches of a lake, alternative to the one given by Proudman (1914 B (2)). Starting with Proudman’s equations he obtained a solution of these by means of an integral equation of the second type due to Volterra.

It is only possible for the current to be everywhere longitudinal when the depth is uniform and the sides strictly parallel. To examine the effect of variable depth, Proudman considered free progressive wave motion of prescribed period in a long channel whose section is uniform along its length but of varying depth from side to side. After making general deductions he examined the case of a channel of parabolic cross-section and, taking account of the transverse currents, compared the results with those obtained by the "narrow sea theory," in which the transverse currents are neglected. He found that if the channel is not too wide or too shallow the degree of accuracy of the narrow sea theory is high, but that it decreases as the effect of the earth’s rotation becomes important. The error depends on the ratio of the period of a free transverse oscillation to the period of the actual motion. For the semi-diurnal tides of such basins as the Red Sea and the English Channel, the error is not very serious. Proudman also considered sea-seiches in a basin of uniform depth contained between a straight coast and a parallel sub-marine ridge and also in an elongated rectangular gulf, investigating the amplitude of oscillations generated by a disturbance out at sea. For the gulf he obtained a formula for the end-correction, which includes as a particular case that of Rayleigh introduced by the Japanese investigators.

F. Vercelli published the results of an extensive series of observations in the Red Sea; the harmonic constants for the $M_1$,
S₂, N₂, K₂, K₁, O₁, P₁, M₄ constituents of tidal elevation were
given for a number of stations distributed along the Gulf of Suez
and the Red Sea, together with those of the current through the
1926 A. (3). Strait of Bab-al-Mandeb. Following on this, Defant undertook a new
computation of the theoretical tides of the Red Sea, restricting
himself to the M₂ and K₁ constituents. Again defining the forced
tide to have zero elevation at the mouth of the sea, he concluded
that the tides of the Red Sea are essentially co-oscillations with
the external tides in the Gulf of Aden. This is in contradiction to
the conclusion he derived in 1918. He suggested that friction might
be important in the Gulf of Suez.

1916 A. The seiches of Lake Vetter in Sweden were thoroughly studied
by F. Bergsten on the lines laid down by Chrystal; he applied
also a correction to the calculated value of the unimodal period to
allow for friction.

1926 B. In the same year Stermeck gave the results of a theoretical
determination of the M₂ constituent of the tides of Lake Baikal,
and further mentioned that his theory of the Adriatic tides (1919
A (1)) stood in agreement with the values of the harmonic con-
stants at stations on the Italian coast computed by Crestani. In
the following year he commented on the contradictory conclusions
of Defant in connection with the ratio of the amplitude of the
forced tide of the Red Sea to that of the co-oscillation. By apply-
ing his criterion to Verzelli's results he found that for greater part
of the sea, this ratio is approximately 1:3 for M₂ and 1:4 for S₁.

1927 A. S. Goldstein considered the free longitudinal oscillations in a
stationary narrow lake of elliptic plan in fuller detail than was
given by Jeffreys in 1924. He gave complete solutions, involving
Mathieu-functions, for the first five normal modes of oscillation and
simple approximate formulae for all modes. In comparison with a
uniform rectangular basin of the same length and depth he found
a decrease in the period of oscillation, which tended to vanish as
the order of the mode increased; the amplitude was, in general,
less in the middle and larger at the ends than in the rectangular
canal.

1927 B. Jeffreys examined the more rapid free longitudinal oscillations
in a stationary lake of varying breadth and depth, the variation
being small except possibly near the ends. He showed that the
periods for a lake with square cut ends are the same as for a lake
of uniform breadth and depth for which a tidal wave takes the
same time to travel from end to end. He found the effects of other shaped ends and considered also the case where the lake becomes shallower towards the ends.

It was suggested by Proudman that the earth-tide under a narrow sea could be determined from observations of the water-tide. He gave two methods, both involving step-by-step integration. The first method requires a knowledge of the tidal elevation at frequent intervals along the sea; the current is obtained from the equation

$$\frac{\partial (A u)}{\partial x} + b \frac{\partial \zeta}{\partial t} = 0,$$

and then $\partial (\zeta_0 - \bar{\zeta})/\partial x$ from

$$\frac{\partial u}{\partial t} = -g \frac{\partial}{\partial x} (\zeta + \zeta_0 - \bar{\zeta} - \bar{\zeta}_0).$$

The second method involves the assumption

$$\zeta_0 = h \bar{\zeta}, \quad \bar{\zeta}_0 = k \bar{\zeta},$$

where $h, k$ are constants, and requires the determination of two sets of functions which, when combined, satisfy (5) and (6). The harmonic constants for two stations would suffice to determine $h, k$. He suggested that the Red Sea would be appropriate to utilise for the purpose.

A. C. Banerji and R. S. Varma considered the case of a canal open to the sea at one end, whose breadth and mean depth vary slowly according to the laws

$$b = b_o (1 + p \cos \nu x), \quad h = h_o (1 + q \cos \nu x),$$

where $p, q$ are small constants and $m, n$ are integers. They obtained a solution for the co-oscillation with the outer tidal motion.

An error in Blondel’s calculations on the tides of the Red Sea (1912 A) was detected by Madame E. Chandon. Its correction removed the greater part of the discrepancy between his theory and the observations. She re-examined the problem, replacing the actual basin by a series of uniform canals and obtained a distribution of amplitude for the $M_2$ tide which compared favourably with actuality over the greater part of the sea, but she failed to secure agreement at Porim.

Sterneck published the results of a theoretical examination of the tides of Lake Baikal, considering in particular the constituents
$M_2$ and $K_2$. Observational data existed only for one station. For this he obtained good agreement in phase but his theoretical amplitudes were nearly twice as large as those derived from observation. He attributed the discrepancy to frictional influences.

The effect of frictional influences on the co-oscillation of a canal with the outer tidal motion was examined by Defant after the manner of Airy. On account of such influences in the Gulf of Suez, he suggested that Stern's criterion for the ratio of co-oscillation to forced oscillation in the Red Sea was invalid, but did not work out numerical details.

In the narrow sea theory the cotidal lines are assumed to be perpendicular to the medial line, apart from the effects of the earth's rotation. On account of this, Proudman examined the possibility of the curvature of the cotidal lines across a long stationary channel of uniform longitudinal section but of varying depth from side to side. He considered the free progression of a tidal wave, defined by the conditions at one end, and examined the particular case in which the depth is uniform on each side of the mid-channel but different on the two sides. He found that for a channel whose breadth is a small fraction of the wave-length, the curvature is very small.

6. STATE OF THE SUBJECT IN 1929.

Up to the end of 1928 the methods of the "narrow sea theory", discussed in § 4 above, had not been applied to the basins of the Bay of Fundy or the Gulf of California, in spite of the suitability of these basins. As regards this theory itself, the treatment of friction had not been made to correspond to the state of knowledge on the subject. Also, the effect of the tidal yielding of the earth's crust had not been allowed for in the detailed explanation of the tides of any actual basin.

The narrow sea theory is in many respects only a first approximation, but in any respect the next approximation for a natural basin presents a difficult problem. It will be noticed from § 5 that second approximations had only been evaluated for geometrically simple basins, though these evaluations indicated that, for such basins as the Adriatic and the Red Sea, the errors of the first approximation can only be small.